Exercise 17

Use the method of undetermined coefficients to find the particular solution for the following initial value problems:

$$u'' - u' = 6, \quad u(0) = 3,\ u'(0) = 2$$

Solution

This is an inhomogeneous ODE, so the general solution is the sum of the complementary and particular solutions.

$$u = u_c + u_p$$

The complementary solution is the solution to the associated homogeneous equation,

$$u_c'' - u_c' = 0.$$

This is a linear ODE with constant coefficients, so the solution will be of the form $u_c = e^{rx}$.

$$u_c = e^{rx} \quad \to \quad u_c' = re^{rx} \quad \to \quad u_c'' = r^2 e^{rx}$$

Substituting these into the equation gives us

$$r^2 e^{rx} - re^{rx} = 0.$$

Divide both sides by e^{rx}.

$$r^2 - r = 0$$

Factor the left side.

$$r(r-1) = 0$$

$r = 0$ or $r = 1$, so the complementary solution is

$$u_c(x) = C_1 + C_2 e^x.$$

Now we turn our attention to the particular solution. Because the inhomogeneous term, 6, is a constant, the particular solution should be chosen so that the higher derivatives vanish but the smallest derivative remains, i.e. $u_p = Ax$. Plugging this form into the ODE yields $-A = 6$, which means $A = -6$. Thus, $u_p = -6x$. Therefore, the general solution to the ODE is

$$u(x) = C_1 + C_2 e^x - 6x.$$

These constants can be determined since initial conditions are given.

$$u'(x) = C_2 e^x - 6$$

$$\begin{aligned} u(0) &= C_1 + C_2 = 3 \\ u'(0) &= C_2 - 6 = 2 \end{aligned}$$

The solution to this system of equations is $C_1 = -5$ and $C_2 = 8$. Therefore,

$$u(x) = 8e^x - 6x - 5.$$

We can check that this is the solution. The first and second derivatives are

$$\begin{aligned} u' &= 8e^x - 6 \\ u'' &= 8e^x. \end{aligned}$$

Hence,

$$u'' - u' = \cancel{8e^x} - (\cancel{8e^x} - 6) = 6,$$

which means this is the correct solution.